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On Plane Cubics

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VII. *On Plane Cubics.*

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No systematic investigation by simple geometrical methods of the variation of the Hessian and Cayleyan as dependent on the variation of the fundamental cubic appears to have been undertaken hitherto, though the general relation of the three curves has been thoroughly studied both geometrically and analytically. This investigation however appears desirable, not only for itself, but also for the sake of the explanation it offers of the importance and interest of some special cubics.

In the following pages the first few sections are devoted to certain constructions for the three curves, which are then applied to special cubics, among these the equianharmonic cubic, whose known properties present themselves very simply by means of the preliminary constructions. The cubics here considered are, as appears in the next section, the critical ones when we follow out the variation of the Hessian and Cayleyan. In conclusion, the results are compared with those derived by analysis, and are exhibited graphically by means of a single diagram.

I. *Construction of the Cubic, its Hessian and Cayleyan.* Figs. 1–3.

1. Let three collinear inflexions of a cubic be I_1, I_2, I_3 (fig. 1); call the intersections of the tangents at these inflexions D_1, D_2, D_3, the points in which they meet the harmonic polars T_1, T_2, T_3, the points in which the harmonic polars T_1D_1, T_2D_2, T_3D_3, *i.e.*, h_1, h_2, h_3 meet the line of inflexions H_1, H_2, H_3, and the intersection of the harmonic polars O, so that O and the line (I) are pole and polar with regard to the triangle $D_1D_2D_3$.

Let the points of contact of the three tangents from I_1, which are necessarily on the harmonic polar h_1, be K_1, k_1, κ_1, &c. The arrangement of the K's is determined by a consideration of the sixteen lines that have (I) for satellite. These sixteen lines are

(1.) $I_1I_2I_3$.

(2.) I_1K_2, which must pass through one of the three points K_3, k_3, κ_3; call this point K_3, and similarly select K_1 by means of I_3K_2; then will $I_2K_1K_3$ be collinear. For $\{I_1I_2H_1I_3\}$ is harmonic, as also $\{I_1K_3V_1K_2\}$, V_1 being the point in which $I_1K_3K_2$ meets the harmonic polar h_1; hence I_2K_3, I_3K_2 must meet on H_1V_1, *i.e.*, on h_1, and therefore necessarily at K_1. Similarly the three points k_1, k_2, k_3 are grouped, and also the remaining three κ_1, κ_2, κ_3, thus giving nine of the sixteen lines.

(3.) For the remaining six; K_1k_2 must go through one of the points on h_3; now this cannot be K_3 or k_3, hence it must be κ_3; thus these six lines are of the type $K_1k_2\kappa_3$.

Now let the tangents at K_2, K_3 meet at G_1, which, by harmonic symmetry, is of course on h_1. We have thus three groups of G's, viz. :—G_1, G_2, G_3; g_1, g_2, g_3; γ_1, γ_2, γ_3, arranged in triangles, corresponding to the K's, and, moreover, collinear in threes, again corresponding to the K's. The proof of this last statement depends on a property proved in the next paragraph, that k_1, κ_1 are harmonic with regard to O, G_1; for then

$$\{I_1H_2H_1H_3\} = \{Ok_1G_1\kappa_1\},$$

i.e.,

$$\{O \,.\, I_1H_2H_1H_3\} = \{I_1 \,.\, Ok_1G_1\kappa_1\}.$$

Therefore the three points (OH_2) (I_1k_1), (OH_1) (I_1G_1), (OH_3) $(I_1\kappa_1)$, *i.e.*, g_2, G_1, γ_3, are collinear.

2. The three collinear inflexions with their tangents amount to eight conditions; thus any one of the nine points K completes the determination of the cubic; consequently the two points k, κ, must be determinable from K*; as a matter of fact they present themselves as the foci of a certain involution.

(*a*.) k, κ are harmonic with regard to OG. One of the four poles of the line (I) (fig. 1) with regard to the cubic is O; hence, estimating on the transversal h, we have

$$\frac{1}{OK} + \frac{1}{Ok} + \frac{1}{O\kappa} = \frac{3}{OH} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}.$$

Now consider the triangle GG_2G_3; OG, *i.e.*, h, meets G_2G_3 in K, &c., and K_2K_3 meets G_2G_3 in I, &c., therefore the line (I) is the polar of O with regard to this triangle. Hence, again estimating on the transversal h,

$$\frac{1}{OK} + \frac{2}{OG} = \frac{3}{OH} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}.$$

From (i.) and (ii.),

$$\frac{1}{Ok} + \frac{1}{O\kappa} = \frac{2}{OG},$$

i.e., k, κ are harmonic with regard to OG.

* Points on the three harmonic polars are naturally distinguished by suffixes 1, 2, 3; but as the conclusions are applicable indifferently to the points on any one harmonic polar, though all the constructions start from h_1, the suffix 1 is *in general* dropped in the text, while the suffixes 2, 3 are retained. The points K_0, G_0, in § 4 are special positions of K_1, G_1.

(*b*.) Let IK meet H_2D in α (fig. 1), and let $I_3\alpha$ meet h in Y_1, *i.e.* Y. Then k, κ are harmonic with regard to DY.

By harmonic symmetry, constructing α' by means of H_3D, $I_2\alpha'$ passes through Y; let $H_2\alpha$ meet $I_2\alpha'$ in ϖ, and similarly for ϖ', then $\varpi\varpi'$ passes through I; hence the quadrilateral $\varpi\varpi' I_2 I_3$ has I, Y for two of its vertices. We have to show that $I_3\varpi$, $I_2\varpi'$, which by harmonic symmetry meet on h, actually meet at K.

We have

$$\{I.HY\varpi'K\} = \{I_3Y\varpi'\alpha\} = \{I_3I_2H_3I_1\}$$

[by projection through α' on to the line (I)], and is therefore harmonic; *i.e.*, K is the intersection of the diagonals.

Now consider the triangle $Y\alpha\alpha'$, and determine the polar of D. Y, α, α', projected through D on to the sides, give K, ϖ, ϖ'; $\varpi\varpi'$, $\varpi'K$, $K\varpi$ meet $\alpha\alpha'$, $\alpha'Y$, $Y\alpha$ at I_1, I_2, I_3; hence the line (I) is the polar of D, and estimating on the transversal h, we have

$$\frac{1}{DK} + \frac{2}{DY} = \frac{3}{DH} \quad \text{(iii.)}.$$

Now the line (I) is the polar of D with regard to the cubic, and therefore

$$\frac{1}{DK} + \frac{1}{Dk} + \frac{1}{D\kappa} = \frac{3}{DH} \quad \text{(iv.)}.$$

From (iii.) and (iv.),

$$\frac{1}{Dk} + \frac{1}{D\kappa} = \frac{2}{DY},$$

i.e., k, κ are harmonic with regard to DY. Thus k, κ are the foci of the involution OG, DY, and are therefore given when K is given.

3. Now the IDH scheme depends on a triangle and one other straight line. Thus any two such schemes can be projected into one another; *i.e.*, excluding for the present (1) the cubic with three real concurrent inflexional tangents, (2) the crunodal cubic, (3) the cuspidal cubic, we may say "all cubics have the same framework." But in connecting projectively the frameworks of two cubics we have exhausted the possibilities of projection, and so have no means of bringing the K's of the two cubics to coincidence; thus different positions of the three K's on h give essentially distinct cubics, so exhibiting clearly the known fact that the essential nature of the general cubic depends on one parameter only.

Since we can project so that the triangle $D_1D_2D_3$ becomes equilateral, while the line (I) goes to infinity, we can always use a symmetrical diagram. This simplification is adopted for most of the diagrams here given.

4. The two points k, κ, will be real or imaginary according to the position of K; they will coincide, so giving the acnodal cubic, when Y comes at O, *i.e.*, when $I_3\alpha$ goes through O. Thus the position of K for the acnodal cubic is the intersection of h with IJ, where J is the intersection of I_3O, H_2D; call this point K_0. If now we take K a very little further away from D, Y is no longer at O, but is between O and T; thus the involution OG, DY, being overlapping, has imaginary foci, and the cubic is unipartite; and similarly taking K a little nearer to D, we see that the cubic is bipartite.

Now suppose that

K travels from K_0 towards H,

then

Y travels from O through T towards H,

and

G travels from G_0 towards H.

Thus G is initially beyond Y (estimating from O on the symmetrical diagram) (fig. 2), and travels at the same rate as G_2, which travels at the same rate as α, and therefore at the same rate as Y; consequently G remains beyond Y, *i.e.*, the involution remains overlapping, and the foci are imaginary. Thus when K is anywhere between H and K_0 the cubic is unipartite.

Now let

K travel from K_0 through D, O, T, towards H,

then

Y travels from O through D, . . . towards H,

and

G travels from G_0 through . . O, D, towards H.

The cubic is initially bipartite, and the segments OG, DY keep clear of one another until G comes at D, *i.e.*, until K is at T; thus the cubic is bipartite when K is anywhere in K_0OT. Similarly taking K in TH, we see that the cubic is unipartite.

5. We next consider the Hessian and the Cayleyan. The Hessian has the same inflexions and harmonic polars, and passes through T_1, T_2, T_3; let the triangle formed by the inflexional tangents be $B_1B_2B_3$, the sides of this meeting the harmonic polars in P_1, P_2, P_3. We have to determine B and P, which can be done by a linear construction; and t, τ, the remaining points in which h meets the Hessian, are found as the foci of a certain involution. As regards the Cayleyan, we know that T is again a point, and that the harmonic polar h is a cuspidal tangent; we arrive at a linear construction for the cusp S; and z, ζ, the remaining two points in which h meets the Cayleyan, present themselves as the foci of an involution.

6. Both the Hessian and the Cayleyan are explicitly dependent on the system of

conic polars, which is constructed from three independent ones. The collinear inflexions give three known conic polars, but these being syzygetic, amount only to two independent ones, leaving one to be determined; the one that is most easily found is the conic polar of K; let this meet h in K′. Since the conic polar of a point on a cubic divides any chord through this point harmonically, K, K′ are harmonic with regard to $k\kappa$, and are therefore conjugate in the involution OG, DY; K′ is therefore determinable by a linear construction as follows:—

By harmonic symmetry, $H_3\alpha$, $H_2\alpha'$ meet on h, at ϵ (figs. 1 and 2). Consider the triangles αDH_3, OG_2K; DH_3, $H_3\alpha$, αD meet G_2K, KO, OG_2 in α', ϵ, H_2, three collinear points; the triangles are therefore in perspective, and αO, DG_2, H_3K meet in a point β; by means of the quadrilateral $I_3G_2\alpha\beta$ we see that $I_3\beta$ determines the conjugate to K in the involution OG, DY. K′ is shown in fig. 2.

7. Now I, T being conjugate poles, we know that t, τ are also conjugate poles, and are therefore conjugate with regard to every conic polar; t, τ are thus conjugate with regard to KK′, and also with regard to OD (since the conic polar of I_2 is the line pair T_2D, T_2O), *i.e.*, t, τ are the foci of the involution OD, KK′.

8. For a certain choice of K, $I_3\beta$ will go through O, *i.e.*, K′ will come at O, and then t, τ coincide, at O; but $I_3\beta$ can go through D only if G_2 be at D_2, which makes K come at T, an impossible arrangement unless the cubic, and therefore also the Hessian, should degenerate; [or if K be at D, which has the same effect.] Thus the Hessian has a double point when $I_3\beta$ goes through O, *i.e.*, when $I_3\alpha\beta O$ are collinear, *i.e.*, when α is the intersection of I_3O and H_2D, the condition already found for the occurrence of a double point on the cubic. Now when K is in the segment TH, K′ is in DH; when K is in HK_0, K′ is in HTO; the foci of OD, KK′ are real, and to the unipartite cubic corresponds a bipartite Hessian. When K is in K_0D, K′ is in OD; when K is in DO, K′ is in DHO; and when K is in OT, K′ is in OD; thus the bipartite cubic gives a unipartite Hessian; and for both cubic and Hessian, the transition from the one form to the other takes place through the nodal form.

9. As regards the Cayleyan, the cusp which has h as a tangent being at S, we know by the ordinary construction for the point of contact of a tangent to the Cayleyan that T, S are harmonic with regard to $t\tau$, and are therefore conjugate in the involution OD, KK′. Let I_3T meet DG_2 in ∂ (figs. 2, 3), and let I_3O meet ∂K in η; by means of the quadrilateral $I_3\beta\partial\eta$ we see that $\beta\eta$ goes through S.

10. The inflexional tangent to the Hessian is determined when S is known; let JS meet IT in λ (fig. 3), then λH_2 goes through B. For the proof of this compare the Hessian, *quâ* cubic, with the original cubic, and apply to it the properties of the diagram for the cubic; for comparison, points on the Hessian may for the moment be denoted by the same letters as corresponding points on the cubic, accented.

We found that k, κ must be the foci of OG, DY, and therefore t, τ are the foci of OG′, D′Y′. But K′ and G′ are respectively T and D, therefore t, τ are the foci of OD, D′Y′; also they are known to be harmonic to TS. Now in the original cubic (fig. 1), H_2D, I_3Y meet on the tangent at K; hence, referring this to the Hessian, H_2D', I_3Y' meet on the tangent at T, *i.e.*, on IT; call their point of meeting λ (fig. 3); we have to determine λ. Since OD, D′Y′, TS are in involution,

$$\{D'ODT\} = \{Y'DOS\}.$$

Project the left-hand side through H_2, and the right-hand side through I_3, on to IT, we then obtain (the points M, N, ρ being as shown in fig. 3)

$$\{\lambda D_2 MT\} = \{\lambda D_2 N\rho\},$$

i.e.,

$$\{\lambda D_2 MT\} = \{D_2 \lambda\rho N\};$$

therefore λ, D_2 are conjugate in the involution $M\rho$, NT. Hence by means of the quadrilateral I_3DJS, we see that JS goes through λ; and then λH_2 goes through D′, *i.e.*, through B. Thus the inflexional tangents to the Hessian are found. A more convenient construction may be deduced; from the identity

$$\{\lambda D_2 TM\} = \{D_2 \lambda MT\},$$

there follows, by projection on to h from H_2 and J,

$$\{BOTD\} = \{QSDT\},$$

i.e., BQ, OS, DT are in involution. Thus to find B, let I_3T meet H_3S in μ; then by means of the quadrilateral $H_3T_2L_2\mu$, we see that $L_2\mu$ goes through B.

11. The points z, ζ on the Cayleyan are its points of contact with the conic polar of T. Now the inflexional tangent to the Hessian, *i.e.*, IP, is known to be the line polar of T with regard to the original cubic; it is therefore the line polar of T with regard to the conic polar of T; and consequently T, P are harmonic with regard to $z\zeta$. Also I_2, T_2 are conjugate poles, and are therefore conjugate with regard to the conic polar we are considering, viz., with regard to Iz, Iζ; therefore projecting from I on to h (fig. 3), we see that W, H are conjugate with regard to $z\zeta$. Thus z, ζ are the foci of the involution TP, WH.

12. The constructions are therefore :—

(1.) IK meets H_2D in α; $I_3\alpha$ meets h in Y; k, κ are the foci of OG, DY (fig. 2.)

(2.) $O\alpha$, DG_2, H_3K meet in β; $I_3\beta$ meets h in K'; t, τ are the foci of OD, KK'.
(3.) I_3T meets DG_2 in ∂; I_3O meets ∂K in η; $\beta\eta$ goes through S (figs. 2, 3).
(4.) I_3T meets H_3S in μ; L_2 is the intersection of H_3D with h_2; $L_2\mu$ goes through B.
(5.) z, ζ are the foci of TP, WH (fig. 3).

13. Now z, ζ being the foci of the involution TP, WH, will be imaginary if P lie in the segment WDH, otherwise real. When P is at W, B is at T; and as P travels over WDH, B travels in the opposite direction over TH. Thus the Cayleyan is unipartite when B is in the segment TH, otherwise it is bipartite. Now when B is in TH, λ (fig. 3) is in TD_3I; S is therefore in THK_0; and when B is in TDH, λ is in TD_2I, and S is in TDK_0. Thus the Cayleyan changes from unipartite to bipartite and *vice versâ* when the cusp passes through T and K_0; but of these two, in the series here considered, K_0 corresponds to the case $K \equiv H$, which gives a degenerate cubic.

II. *Application to Special Cubics.* Figs. 4, 5.

14. *The Harmonic Cubics.*—If the cubic be harmonic, let K be the one of the three points on h that is conjugate to T, *i.e.*, let K, T be harmonic with regard to $k\kappa$. Then since K, K' are harmonic with regard to $k\kappa$, K' now comes at T. In the general case T, S are points in which h meets a series of conic polars; hence, T being K', S must be K; *i.e.*, for a harmonic cubic, the cusps of the Cayleyan are on the cubic. Conversely, if S come at K, K' must come at T, and the cubic is harmonic.

Now in the case we are considering, the conic polar of K goes through T, hence the line polar of T goes through K; *i.e.*, the inflexional tangent to the Hessian goes through K; thus P is at K. Conversely, if P be at K, *i.e.*, if the line polar of T pass through K, then the conic polar of K passes through T, thus K' is at T, and as before, the cubic is harmonic.

In the general case, t, τ are harmonic with regard to KK', and therefore in this case with regard to TK, *i.e.*, with regard to TP; hence the Hessian is harmonic; and as z, ζ are harmonic with regard to TP, *i.e.*, with regard to TS, the Cayleyan, *quâ* class-cubic, is also harmonic.

The question now is, where must K be in order that the cubic may be harmonic.

When S comes at K, H_3S coincides with H_3K, therefore μ is on $K\beta$; η is also on $K\beta$, since $\beta\eta$ has to go through S, likewise ∂, since $\partial\eta$ goes through K. But $\beta\partial$ goes through D, hence ∂ must be at β; and since $\partial\mu$ goes through I_3, μ and ∂ must coincide at β.

The pencils $\{T_2 . G_2\beta DW\}$, $\{K . G_2\beta DH_2\}$ (fig. 4) estimated on the line (I) are equal to

$$\{H_2I_3I_2I_1\} \quad \text{and} \quad \{I_1H_3H_1H_2\} \text{ respectively;}$$

but these are equal, and therefore

$$\{T_2 . G_2\beta DW\} = \{K . G_2\beta DH_2\};$$

hence T_2W, KH_2 must meet on the line $G_2\beta D$, at ϑ.

Projecting {DWOK} from ϑ on to h_2, it becomes $= \{G_2T_2OH_2\}$, which by projection from I on to $h = \{KWOH\}$, therefore

$$\{DWOK\} = \{HOWK\},$$

therefore K is self-conjugate in the involution HD, OW; *i.e.*, for a harmonic cubic the point K is a focus of HD, OW. Hence there are two such cubics, one with K as in fig. 4, giving a unipartite cubic; one with K between O, W, giving a bipartite cubic. These points are at once found in the symmetrical diagram; for H being at infinity, D is the centre of the involution; and since $DT_2^2 = DW . DO$, we must have $DK = DT_2$. Thus the two positions of K are as in figs. 8, 12.

15. *The Equianharmonic Cubics.*—In special cases three inflexional tangents may be concurrent, this being allowed by the class of the cubic being $= 6$; but not more than three. Further, the three will be tangents at *collinear* inflexions; for the line polar of the intersection of two inflexional tangents is the join of the inflexions, and thus if a third inflexional tangent pass through this point, the third inflexion must be the one that lies on this line. We can certainly find a line of inflexions for which the tangents are not concurrent, and therefore if we disregard the distinction between real and imaginary, we can still use the symmetrical triangular diagram; the three concurrent tangents cannot meet in O (for the polar line of O is the line (I), which joins inflexions having non-concurrent tangents), therefore by triangular symmetry there must be *three* sets of concurrent tangents; plainly if one of these be composed of the three real tangents, the other two must be composed of imaginary ones; in the other possible arrangement, the sets are composed each of one real and two imaginary tangents.

Considering the two tangents that are concurrent with IT, we know that these two, being tangents at inflexions collinear with I, must meet on h; their intersection is therefore at T. Now the Hessian has to touch each of these inflexional tangents, in addition to cutting it at the inflexion; passing through T, it cannot meet the inflexional tangent again so as to touch it, consequently for every one of these three inflexional tangents the "contact" has to be at T; there can therefore only be improper contact, *i.e.*, the Hessian must have a double point at T_1, and similarly at T_2 and T_3; it is therefore composed of the three lines T_2T_3, T_3T_1, T_1T_2. Now we know that the line polar of T is the inflexional tangent to the Hessian at I; and we have seen that, for the case we are considering, the line polar of T is the line joining the inflexions whose tangents are concurrent in T; this line polar is therefore the tangent to the Hessian at each of the three inflexions, *i.e.*, it forms a part of the Hessian. Thus the line T_2T_3 joins three inflexions, and the tangents at these three inflexions pass through T_1; *i.e.*, the Hessian is composed of the three lines joining the inflexions whose tangents are concurrent.

Conversely, if the Hessian be composed of three straight lines, the inflexional tangents to the cubic (if a proper cubic) are concurrent in threes. For these nine inflexional tangents have to "touch" the Hessian; they must therefore have improper contact, *i.e.*, they must pass through the three double points T_1, T_2, T_3 of the Hessian, and there being nine of them, three must go through each point T.

Thus the two conditions, "the inflexional tangents are concurrent in threes," and "the Hessian is three straight lines" are coextensive; and there is plainly no need to exclude the degenerate cubics from this enunciation.

The two points t, τ now come at T, W; therefore T, W are the foci of OD, KK′, TS; *i.e.*, S must come at T, and therefore the Cayleyan is composed of the three points T. For P is at W, therefore z, ζ are the foci of an involution which degenerates into TW, WH, *i.e.*, they are at W, and consequently double points and double tangents (at W) are introduced on the Cayleyan. But it has already its maximum number, and therefore it is now a degenerate curve. Being a class-cubic, and preserving its triangular symmetry while degenerating so as still to pass through T_1, T_2, T_3, it can only degenerate into these three points.

Conversely, if the Cayleyan split up into three points, since the cusps cannot disappear, and the points T are in all cases points on the Cayleyan, we know that the three points are the points T, and that the degeneration is brought about by the coincidence of S with T. Now T, S have been proved conjugate in OD, KK′, hence in this case T is a focus of OD, KK′; but t, τ are the foci of this involution, and therefore one of the two points t, τ, must come at T; and thus the Hessian has a double point at T_1, and similarly at T_2 and T_3.

The condition therefore that "the Cayleyan splits up into three points" is equivalent to those already discussed.

We have now to show that if three inflexional tangents be concurrent, the cubic is equianharmonic. Referring the diagram to the concurrent tangents, α comes at G_2, Y at G, and thus the construction requires modification. In the general case T, I, and therefore in the present case O, I, are conjugate poles on the Hessian, and are therefore conjugate with regard to any conic polar; similarly for O, I_2 and for O, I_3. Thus the line (I) is the polar of O with regard to every conic polar; *i.e.*, O, H are conjugate with regard to the conic polar of K, and therefore with regard to K K′; thus K′ is known.

Now {KGHO} by projection through G_3 (fig. 5)

$$= \{I_1I_2H_1H_3\}$$
$$= \{I_2I_1H_3H_1\};$$

and as KK′ are harmonic with regard to OH, X in the equation

$$\{KGHOK'\} = \{I_2I_1H_3H_1X\}$$

must be such that I_2X may be harmonic with regard to H_1H_3; *i.e.*, X must be H_2, therefore

$$\{KGHOK'\} = \{I_2I_1H_3H_1H_2\};$$

therefore

$$\{KGOK'\} = \{I_2I_1H_1H_2\}.$$

Now the foci of the involution OG, KK′, are k, κ; call the foci of I_1H_1, I_2H_2, I_3H_3, x, x'; from the relation just proved we have

$$\{GOKK'k\kappa\} = \{I_1H_1I_2H_2xx'\}.$$

We wish to prove $\{OKk\kappa\}$ equianharmonic; *i.e.*, we have to prove $\{H_1I_2xx'\}$ equianharmonic, for which it suffices to show

$$\{I_2H_1xx'\} = \{I_2x'H_1x\}.$$

Consider the IH involution, whose foci are xx'. From the way it is constructed (viz., three points I, their harmonic conjugates H), we know that any cross-ratio in the I's and H's is unaltered

(1) by any interchange of the suffixes,
(2) by the interchange of I and H.

It is convenient to write 1, 1′, for I_1, H_1, &c.

We have to prove

$$\{21'xx'\} = \{2x'1'x\}.$$

We know that xx', 12, 1′2′ are harmonic with regard to 33′, and therefore in involution; therefore

$$\{121'x\} = \{212'x'\} = \{2'x'21\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i).$$

Now $\{11'xx'\}$ is harmonic, as also $\{2'213\}$; applying these to (i) we have

$$\{121'xx'\} = \{2'x'213\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii).$$

Again, $\{121'3\}$ is harmonic, as also $\{2'x'2x\}$; applying these to (ii) we have

$$\{121'xx'3\} = \{2'x'213x\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii).$$

from which

$$\{12x3\} = \{2'x'1x\},$$

i.e.,

$$\{x123\} = \{12'x'x\} \quad \text{(iv).}$$

Similarly

$$\{212'xx'3\} = \{1'x'123x\} \quad \text{(v),}$$

and therefore

$$\{x123\} = \{2x'1'x\} \quad \text{(vi).}$$

From (iv) and (vi)

$$\{12'x'x\} = \{2x'1'x\} \quad \text{(vii).}$$

Now since xx' are the foci of $11'$, $22'$, we have

$$\{12'x'x\} = \{1'2x'x\},$$

therefore (vii) becomes

$$\{1'2x'x\} = \{2x'1'x\},$$

i.e.,

$$\{21'xx'\} = \{2x'1'x\},$$

i.e., $\{I_2H_1xx'\}$ is equianharmonic, and therefore $\{OKk\kappa\}$ is equianharmonic; *i.e.*, if three inflexional tangents be concurrent, the cubic is equianharmonic.

Conversely, if the cubic be equianharmonic, the inflexional tangents are concurrent in threes. We know that $\{KK'k\kappa\}$ is harmonic, and therefore

$$= \{I_1H_1I_2I_3\} \quad \text{(viii),}$$

and for this special case $\{TKk\kappa\}$ is equianharmonic, and therefore

$$= \{I_1H_2xx'\} \quad \text{(ix).}$$

Now by (viii) and similar relations,

$$\{Kk\kappa K'k'\kappa'\} = \{1231'2'3'\}, \quad \text{(x).}$$

and t, τ are the foci of the left hand side, x, x' of the right.

By means of (ix.), (v.), and (x.),

$$\begin{aligned}\{TKk\kappa\} &= \{12'xx'\} \\ &= \{x'123\} \\ &= \{\tau Kk\kappa\},\end{aligned}$$

where τ is one of the pair t, τ. Thus one of the two points t, τ comes at T, and

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therefore the Hessian is three straight lines, and the inflexional tangents to the cubic are concurrent in threes.

Now for an equianharmonic cubic, the three points K, k, κ are not differentiated as they are for a harmonic cubic; therefore they cannot be found by linear and quadratic constructions. But plainly they cannot all be real, and the cubic is therefore unipartite.

16. Other special cubics might be considered, as for instance the one for which B and P coincide; this coincidence is necessarily at O, and thus the Hessian is equianharmonic. In the general case, BQ, OS, DT are in involution, thus in this case OQ, OS, DT are in involution, and therefore S comes at Q. Moreover, z, ζ, the foci of TP, WH are now the foci of TO, WH, and are therefore real, giving a bipartite Cayleyan.

Again, the three cusps on the Cayleyan may be collinear, *i.e.*, S may be at H. In this case B is conjugate to Q in the involution OH, DT, and therefore comes at L; and t, τ are now the foci of OD, TH, and are therefore real; thus the Hessian is bipartite. In both these cases K cannot be found by linear or quadratic constructions.

III. *Variation in the Hessian and Cayleyan as the Cubic varies.* Figs. 6–13.

17. The cubics just considered are of interest in studying the variation of the Hessian and Cayleyan as dependent on the variation of the original cubic. Figs. 6–13 exhibit this variation; the cubic is represented by the heavy lines, the Hessian by the faint lines, and the Cayleyan is dotted. For these figures the point K was assigned, and the points k, κ; t, τ; S; B; z, ζ, determined by the constructions of § 12; for figs. 7 and 11 the position of K was determined by approximation and trial.

K starts from D, and describes the segment DHT, the segment TOD being described by the complementary k, κ for the bipartite cubic. The inflexional triangle for the Hessian (fig. 6) is at first turned the same way as that for the original cubic, but then by transition (fig. 7) through the form for which the Hessian is equianharmonic, it turns the other way. The tricusp of the Cayleyan shrinks up, until the cusps, initially outside the oval of the cubic, are on the cubic (fig. 8), which is now harmonic, and accompanied by a unipartite harmonic Hessian. The tricusp is now inside the oval, and both shrink up to the point O, giving the acnodal cubic, with an acnodal Hessian, and a degenerate Cayleyan composed of the point O and a conic, which for the symmetrical diagram is the circle inscribed in the triangle $D_1D_2D_3$. At this stage all trace of the oval is lost, but the oval of the Hessian makes its appearance. The tricusp of the Cayleyan cannot disappear, so it now expands from the point form, reversed in position (fig. 9) as compared with its original form. The cusp and the point B approach T together, and we have the equianharmonic cubic, with degenerate Hessian and Cayleyan. Through the degenerate three-point form the Cayleyan passes from bipartite to unipartite (fig. 10). The cusps recede from T through H towards D, passing through the form for which they are at H (fig. 11), and therefore collinear on the line infinity. After this, we have the unipartite harmonic cubic, with a bipartite harmonic Hessian (fig. 12); the infinite branches of the Hessian are outside the limits of the diagram, but fig. 8 represents, on a smaller scale, the relation of the cubic (fine line) to the Hessian (heavy line). As K still recedes towards H, the cusp approaches K_0; when K reaches H, the series gives a degenerate cubic; but if we substitute for this the one that belongs to the series of proper cubics (see No. 351, in vol. 5, of Professor CAYLEY's collected papers) viz., the one with the real inflexional tangents concurrent,* we have the change as in the case of the other equianharmonic cubic—the Hessian is three straight lines, and the Cayleyan changes from unipartite to bipartite through the three-point form. We then have (fig. 13) the quadrilateral unipartite cubic, with the bipartite Hessian and Cayleyan, these, as K approaches T, tending to coincidence with the sides and vertices of the triangle $D_1D_2D_3$.

IV. *Analytical Expression.* Fig. 14.

18. In considering the appearance of the cubic and its derived curves, the equation

$$(x + y + z)^3 - 6\lambda xyz = 0$$

(discussed and compared with HESSE's form by Professor CAYLEY, *loc. cit.*) appears more convenient than HESSE's canonical form. It postulates only three inflexions, so excluding only the cuspidal form, and is therefore more comprehensive; it relates only to elements all of which may be taken real, except for two special cubics, and is therefore convenient when diagrams are required.

The invariants for this form are

$$S = -\lambda^3(4 - \lambda); \quad T = -8\lambda^4(6 - 6\lambda + \lambda^2);$$
$$\Delta = T^2 - 64S^3 = -4 \times 64 \times \lambda^8(2\lambda - 9);$$

* In order to deal with the cubic whose real inflexional tangents are concurrent, while preserving the distinction between real and imaginary, suppose the lines (I), h_1, h_2, h_3, to remain fixed, while the triangle formed by the inflexional tangents changes, D approaching O and then passing through it, so that the segments ODH, OTH are interchanged. The point K_0 is indefinitely near to O, so K is beyond K_0, and the cubic is unipartite. Let K remain fixed, and let it be initially in the segment ODH, then by the interchange of segments it is finally in OTH; consequently B, initially in OTH, is finally in ODH; and (§ 13) the Cayleyan changes from unipartite to bipartite.

and the "numerical characteristic" k

$$(= 64S^3/T^2) = -\lambda(4-\lambda)^3 / (6-6\lambda+\lambda^2)^2.$$

The cubic is therefore bipartite or unipartite according as $2\lambda - 9$ is positive or negative.

The Hessian is

$$v^3 - 6\mu x'y'z' = 0,$$

where

$$(6-\lambda)\,x' = 2v - \lambda x, \text{ \&c.},$$
$$v = x+y+z = x'+y'+z',$$
$$\mu = (6-\lambda)^3 / 3\,(4-\lambda)^2,$$

therefore

$$2\mu - 9 = -\lambda^2(2\lambda-9) / 3\,(4-\lambda)^2.$$

The inflexional tangents to the Hessian are $2v - \lambda x = 0$, &c.; these are concurrent if $\lambda = 6$; they coincide with T_2T_3, &c., *i.e.*, with $v - 2x = 0$, &c., if $\lambda = 4$.

The Cayleyan is

$$w^3 - 6\rho\xi'\eta'\zeta' = 0,$$

where

$$2\,(3-\lambda)\,\xi' = -w - (2\lambda-9)\,\xi, \text{ \&c.},$$
$$w = \xi+\eta+\zeta = \xi'+\eta'+\zeta',$$
$$\rho = 2\,(3-\lambda)^3 / 3\,(4-\lambda),$$

therefore

$$2\rho - 9 = -\lambda\,(2\lambda-9)^2 / 3\,(4-\lambda).$$

The cusps are $(2\lambda-8)\,\xi+\eta+\zeta = 0$, &c.; *i.e.*, they are at $(2\lambda-8,\ 1,\ 1)$, &c.; they are therefore collinear if $\lambda = 3$; and they are on the inflexional tangents to the Hessian if

$$2\,(2\lambda-6) - \lambda\,(2\lambda-8) = 0,$$

i.e., if

$$\lambda^2 - 6\lambda + 6 = 0\,;$$

thus for the harmonic cubics $\lambda = 3 \pm \sqrt{3}$.

When λ assumes the values 6 (fig. 7), $3+\sqrt{3}$ (fig. 8), 9/2, 4, 3 (fig. 11), $3-\sqrt{3}$ (fig. 12), the numerical characteristic has the values 4/3, ∞, 1, 0, $-1/3$, $-\infty$.

19. The diagrams here given have been made by means of § 12; but from the analytical expressions just quoted a graph can be constructed, by means of which these may be readily drawn, and the variation possibly more easily grasped.

Arranging the coordinates so as to give actual distances, with $x+y+z=1$ for fundamental identical relation, we wish to determine the various points on h, *i.e.*, on $y=z$; we have therefore

$$x+2y=1.$$

For the cubic,

$$1-6\lambda xy^2=0,$$

i.e.,

$$3\lambda x(x-1)^2-2=0 \quad \ldots \ldots \quad (1).$$

For t, τ, points on the Hessian,

$$\lambda(x-1)^2+6(x-1)+2=0 \quad \ldots \ldots \quad (2).$$

For S, the cusp on the Cayleyan,

$$x:y:z=2\lambda-8:1:1;$$

therefore

$$x=\frac{4-\lambda}{3-\lambda},$$

i.e.,

$$(\lambda-3)(x-1)+1=0 \quad \ldots \ldots \quad (3).$$

For B, the intersection of inflexional tangents to the Hessian,

$$2v-\lambda y=0, \quad 2v-\lambda z=0;$$

therefore

$$x=1-\frac{4}{\lambda},$$

i.e.,

$$\lambda(x-1)+4=0 \quad \ldots \ldots \quad (4).$$

For P, the intersection of h with the inflexional tangent to the Hessian,

$$2v-\lambda x=0,$$

i.e.,

$$\lambda x-2=0 \quad \ldots \ldots \quad (4').$$

For z, ζ, points on the Cayleyan, most simply determined as the foci of TP, WH,

$$\lambda x(x-1)+1=0 \quad \ldots \ldots \quad (5).$$

By means of these six curves, all of which can easily be drawn with a considerable degree of accuracy, we have a diagram (fig. 14), in which for any arbitrarily chosen ordinate λ the abscissæ* give the positions of all the points required in constructing the selected cubic, its Hessian, and its Cayleyan. It will be noticed that the curves (P) and ($t\tau$) touch at $x=\frac{1}{2}$, $\lambda=4$; that (K), (P), and (S) meet where $\lambda=3\pm\sqrt{3}$, and that (P) and (B) meet where $\lambda=6$, agreeing with the conclusions of §§ 14–16.

* For the sake of distinctness, in fig. 14 the abscissa x is measured on a scale three times that of the ordinate λ.

NOTE ADDED FEBRUARY 19, 1894.

[It may be proper to give the point equation of the Cayleyan, the cubic being in the form here considered,

$$(x + y + z)^3 - 6\lambda xyz = 0.$$

The line equation of the Cayleyan is

$$(\xi' + \eta' + \zeta')^3 - 6\rho\xi'\eta'\zeta' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}),$$

eliminating ζ' from this and

$$x'\xi' + y'\eta' + z'\zeta' = 0$$

we obtain a cubic equation in $\xi' : \eta'$,

$$\xi'^3\mathrm{Y}^3 + 3\xi'^2\eta'\{\mathrm{XY}^2 + 2\rho x'z'^2\} + 3\xi'\eta'^2\{\mathrm{X}^2\mathrm{Y} + 2\rho y'z'^2\} + \eta'^3\mathrm{X}^3 = 0,$$

where $\mathrm{X} = z' - y'$, $\mathrm{Y} = z' - x'$.

The discriminant of this, equated to zero, gives the reciprocal to (i.).

With the ordinary notation for the coefficients of the cubic equation, the result is

$$a^2d^2 + 4ac^3 - 6abcd + 4b^3d - 3b^2c^2 = 0,$$

which may be written

$$a^2d^2 + ac^3 + db^3 - 3\left(\frac{ad + bc}{2}\right)^2 = 0.$$

Writing for a, b, c, d their values, we have

$$a^2d^2 + ac^3 + db^3 = 3\mathrm{X}^6\mathrm{Y}^6 + 6\rho z'^2\mathrm{X}^4\mathrm{Y}^4(x'\mathrm{X} + y'\mathrm{Y})$$
$$+ 12\rho^2z'^4\mathrm{X}^2\mathrm{Y}^2(x'^2\mathrm{X}^2 + y'^2\mathrm{Y}^2) + 8\rho^3z'^6(x'^3\mathrm{X}^3 + y'^3\mathrm{Y}^3);$$

$$\frac{ad + bc}{2} = \mathrm{X}^3\mathrm{Y}^3 + \rho z'^2\mathrm{XY}(x'\mathrm{X} + y'\mathrm{Y}) + 2\rho^2x'y'z'^4.$$

Substituting, and noticing that

$$x'\mathrm{X} - y'\mathrm{Y} = z'(x' - y'),$$

and that therefore a factor $\rho^2z'^6$ divides out, we have the reciprocal to (i.) in the form

$$9(y'-z')^2(z'-x')^2(x'-y')^2$$
$$+4\rho\{2y'z'-x'(y'+z')\}\{2z'x'-y'(z'+x')\}\{2x'y'-z'(x'+y')\}-12\rho^2x'^2y'^2z'^2=0.$$

Here x', y', z' are the point coordinates associated with ξ', η', ζ'; we have therefore to transform to x, y, z, the original point coordinates.

Since

$$-(2\lambda-9)\xi = w+2(3-\lambda)\xi', \text{ \&c.}$$
$$=(7-2\lambda)\xi'+\eta'+\zeta', \text{ \&c.},$$

the formulæ of transformation for x', y', z' (the inverse substitution) can be written

$$x'=(7-2\lambda)x+y+z, \text{ \&c.}$$
$$=v+2(3-\lambda)x, \text{ \&c.},$$

where

$$v=x+y+z.$$

Hence

$$y'-z'=2(3-\lambda)(y-z), \text{ \&c.}$$

and

$$2y'z'-x'(y'+z')=2(3-\lambda)v\{y+z-2x\}+4(3-\lambda)^2\{2yz-x(y+z)\}, \text{ \&c.}$$

By means of these, and the value of ρ in terms of λ, the point equation of the Cayleyan is found to be

$$108(4-\lambda)^2(y-z)^2(z-x)^2(x-y)^2$$
$$+4(4-\lambda)\{v(y+z-2x)+2(3-\lambda)(2yz-zx-xy)\}\{z, x\}\{x, y\}$$
$$-\{v+2(3-\lambda)x\}^2\{v+2(3-\lambda)y\}^2\{v+2(3-\lambda)z\}^2=0.$$

The agreement of this with equations (3) and (5) of § 19 may be exhibited by writing it in the form

$$(y-z)^2\Phi_4-(9-2\lambda)^2x\{x+(4-\lambda)(y+z)\}^3\{x^2+(2-\lambda)x(y+z)+(y+z)^2\}=0,$$

which shows that there is a cusp, tangent to $y-z=0$, at the intersection of $y-z=0$ and $x+(4-\lambda)(y+z)=0$, *i.e.*, at $x+(4-\lambda)(1-x)=0$, *i.e.*, at $(\lambda-3)(x-1)+1=0$ (3); and that the line $y-z=0$ also meets the curve on $x=0$ and on the two lines $x^2+(2-\lambda)x(y+z)+(y+z)^2=0$; *i.e.*, at $y=z$, $x^2+(2-\lambda)x(1-x)+(1-x)^2=0$; which last reduces to

$$\lambda x^2-\lambda x+1=0 \text{ (5).]}$$

Fig. 1.

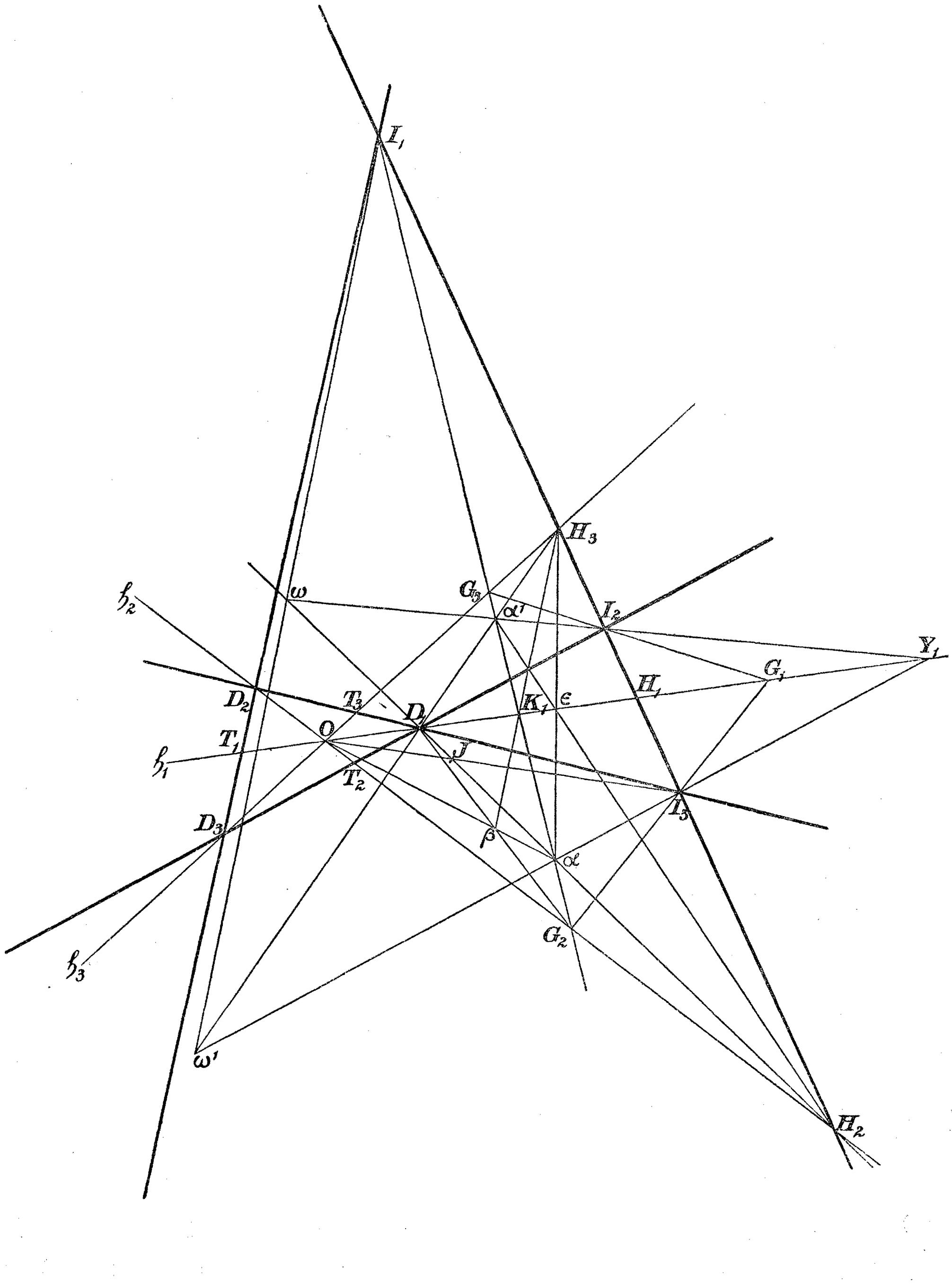

Fig. 2.

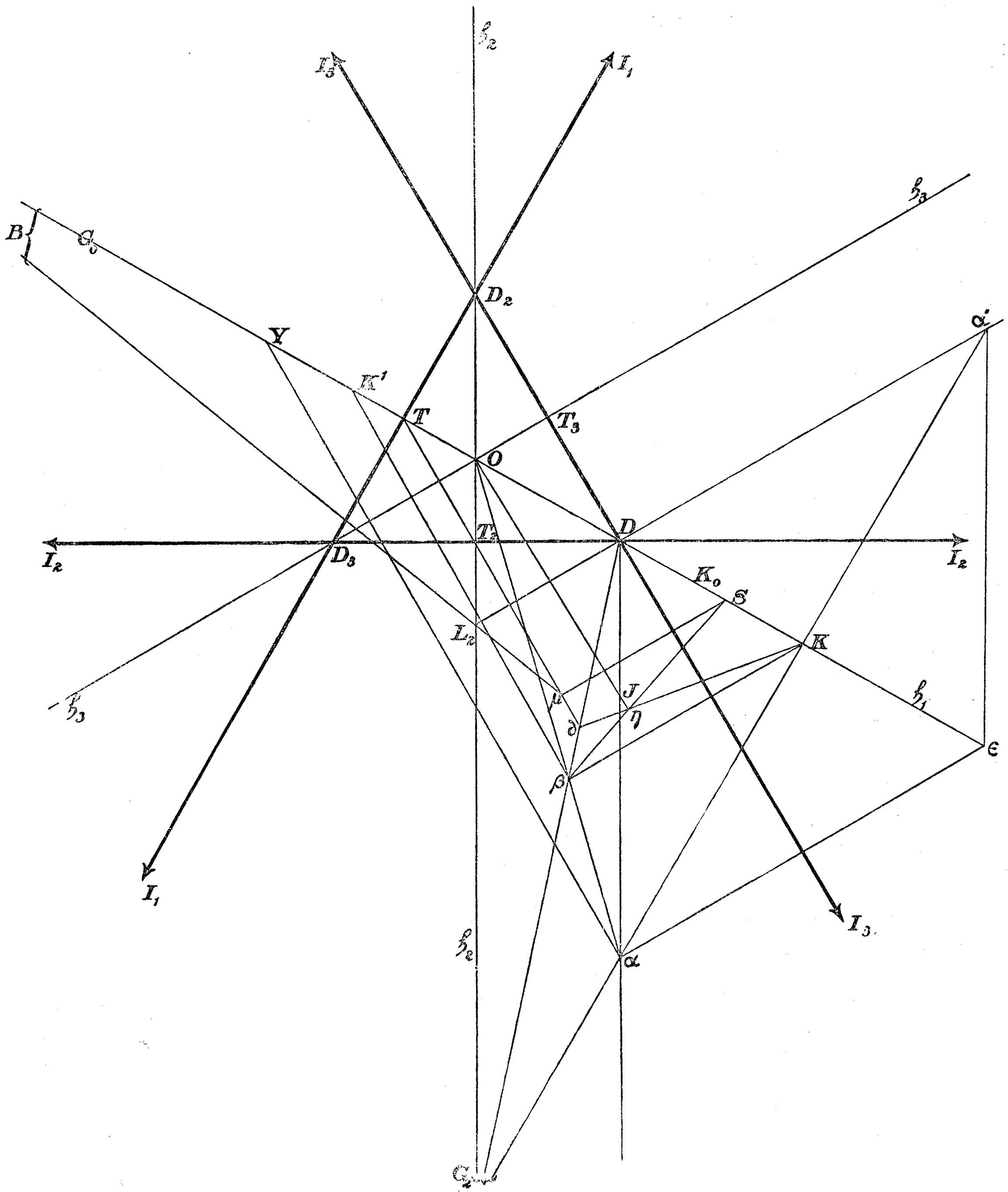

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Fig. 3.

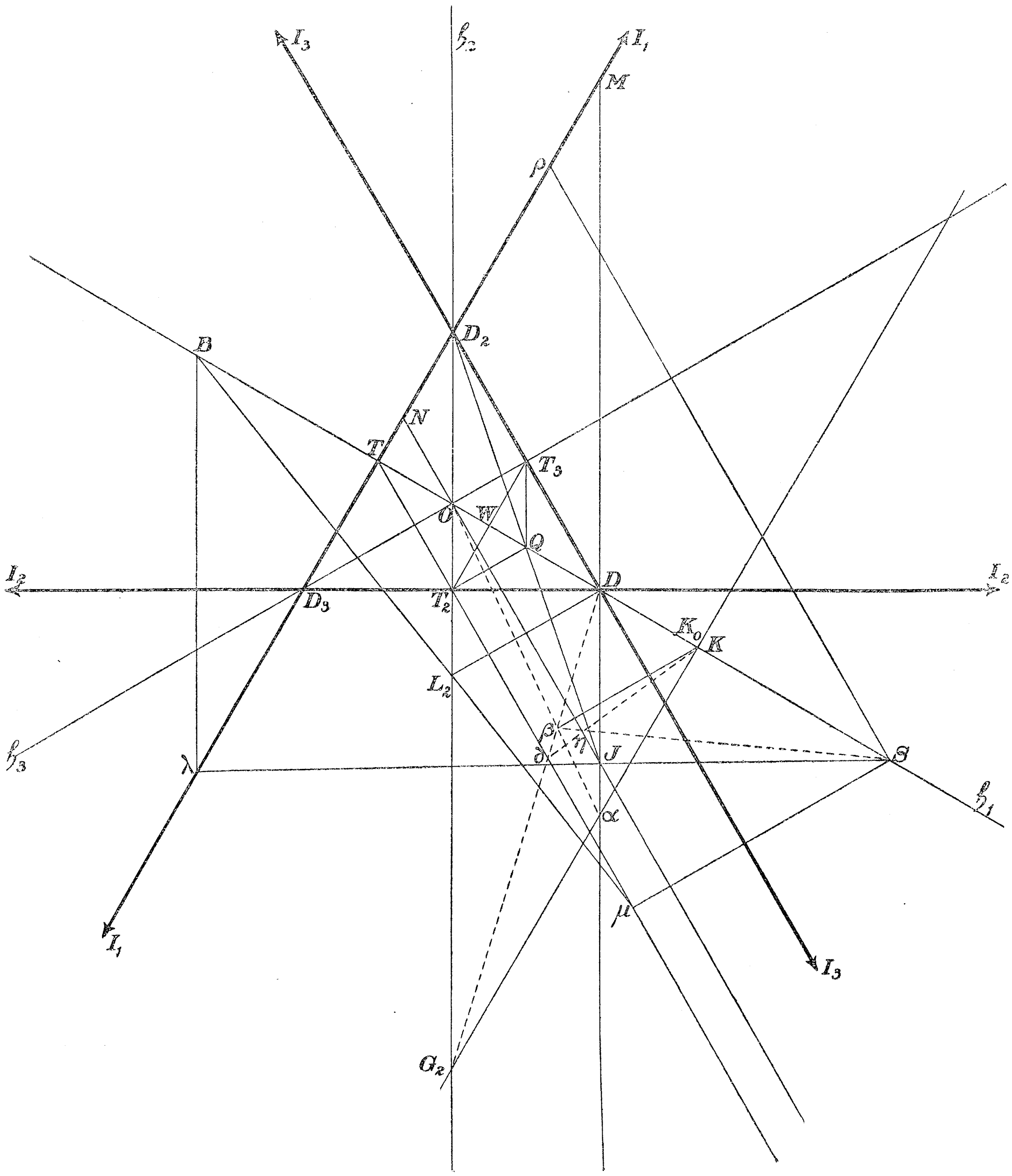

Fig. 4.

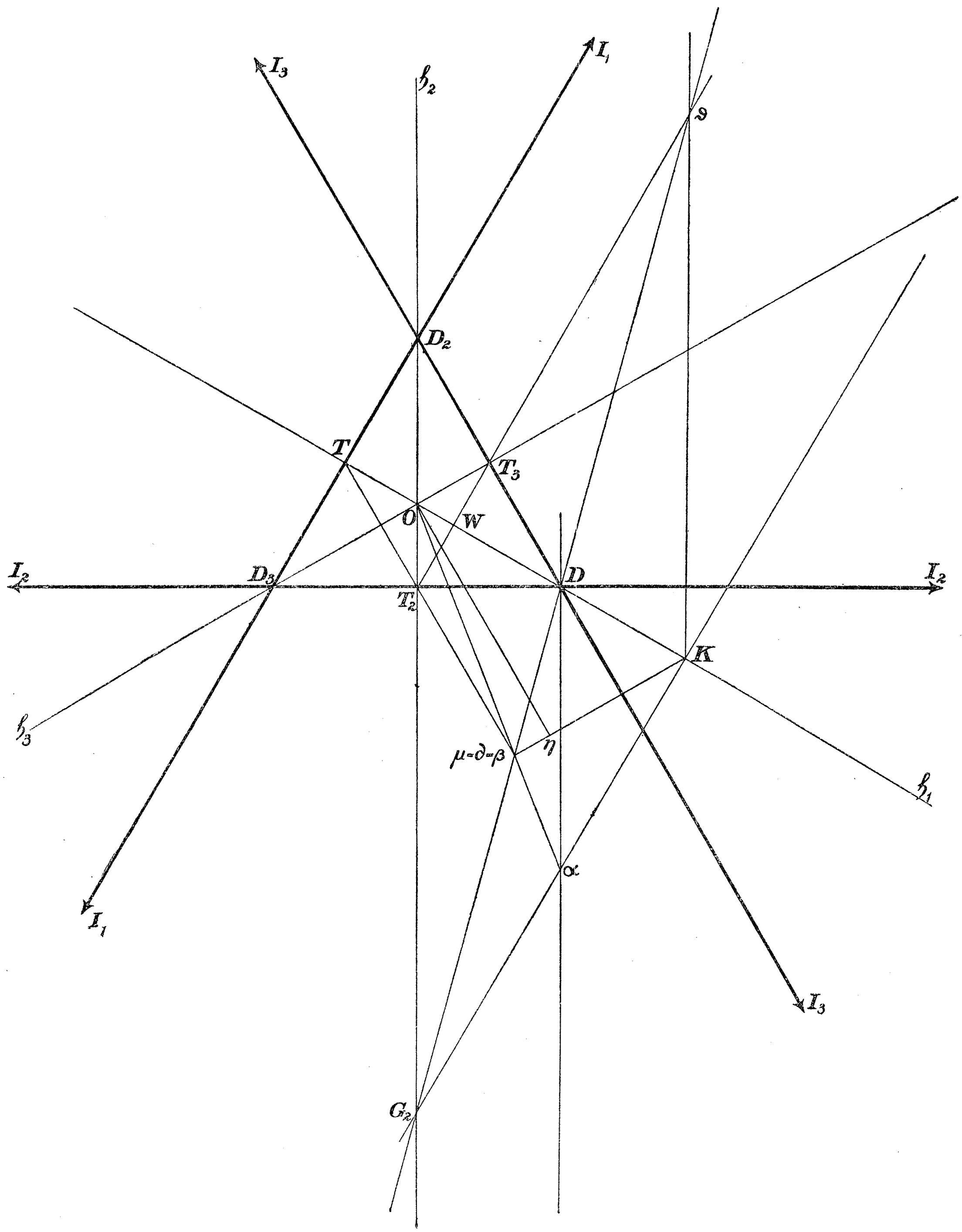

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Fig. 5.

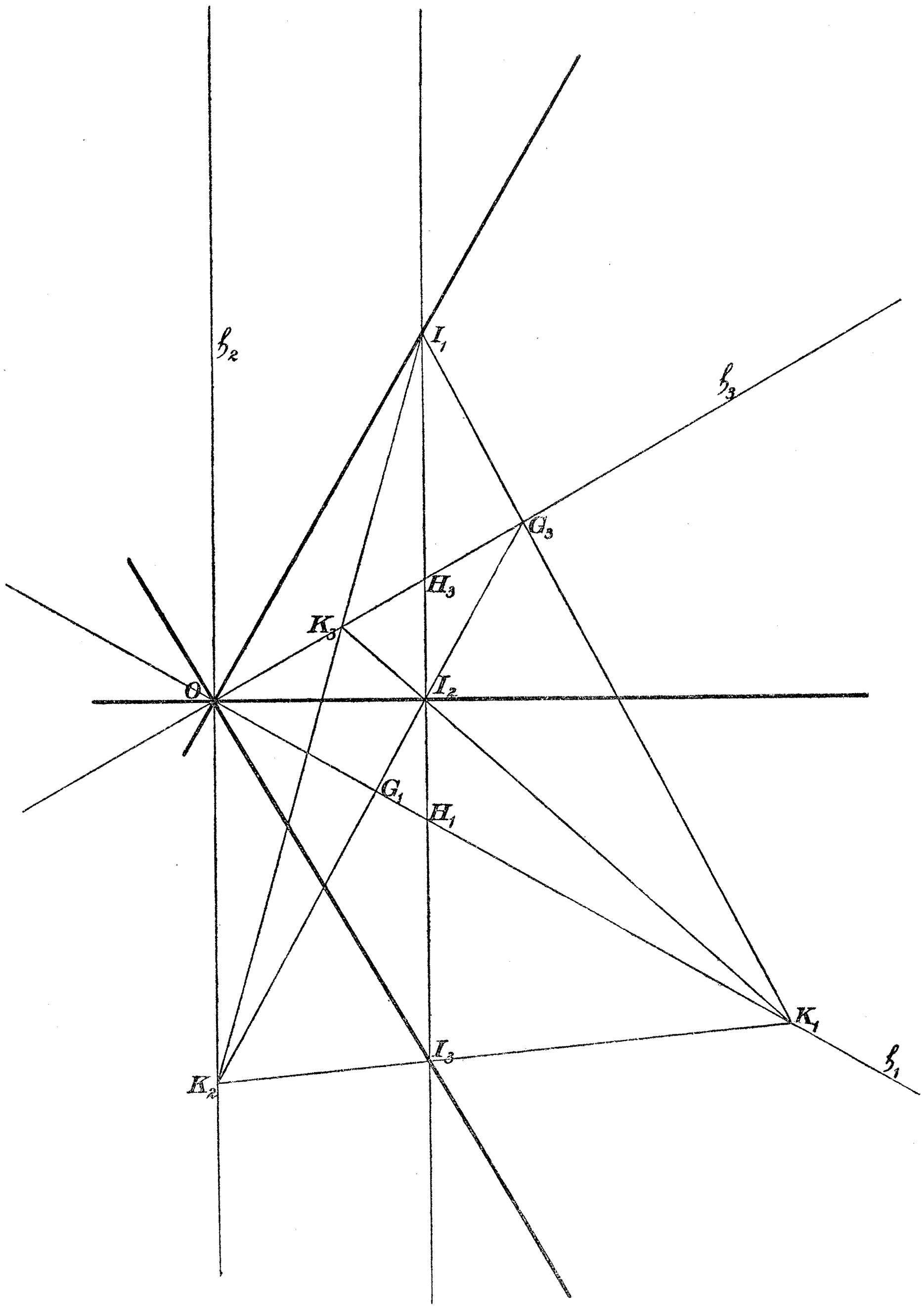

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Fig. 6.

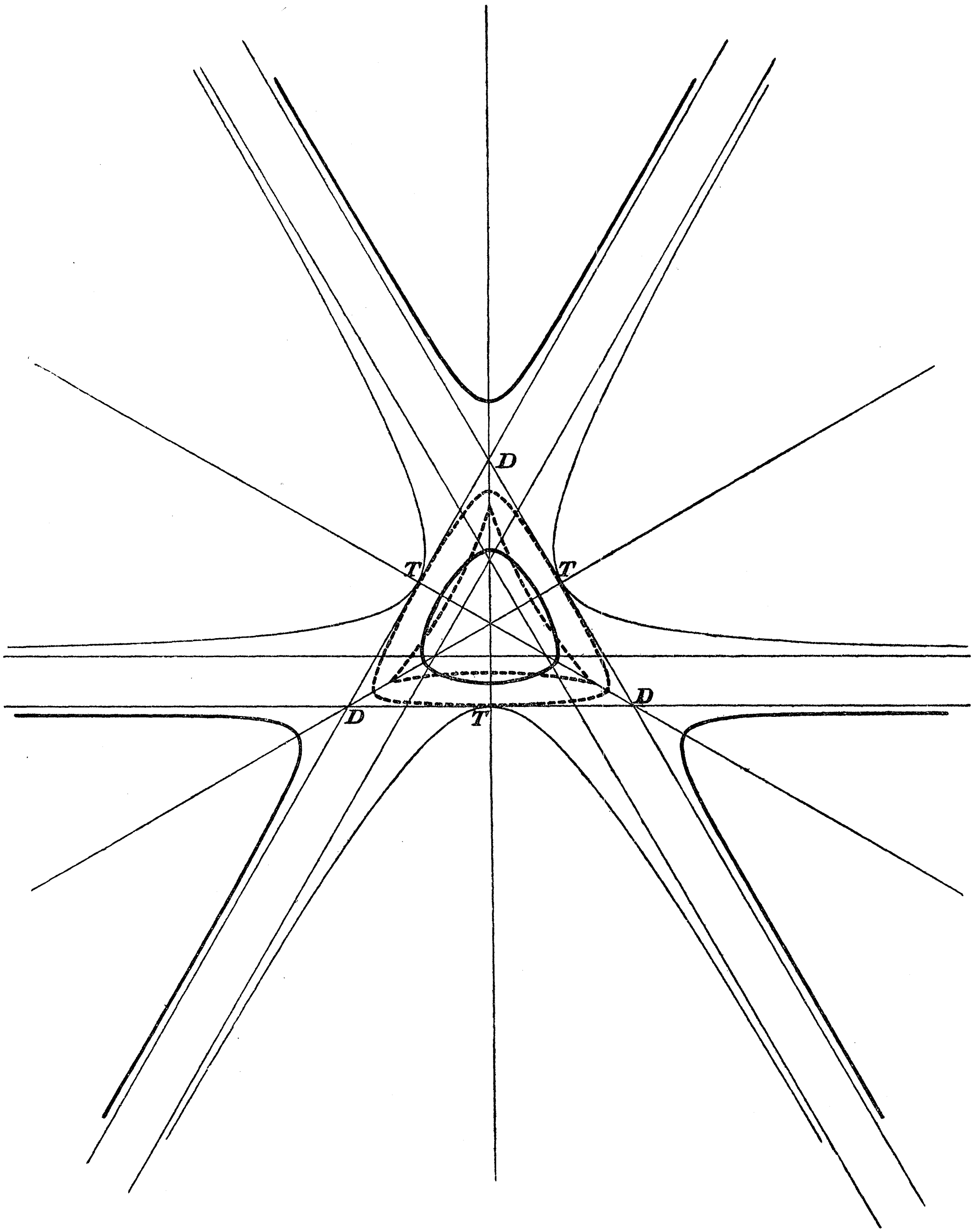

Fig. 7.

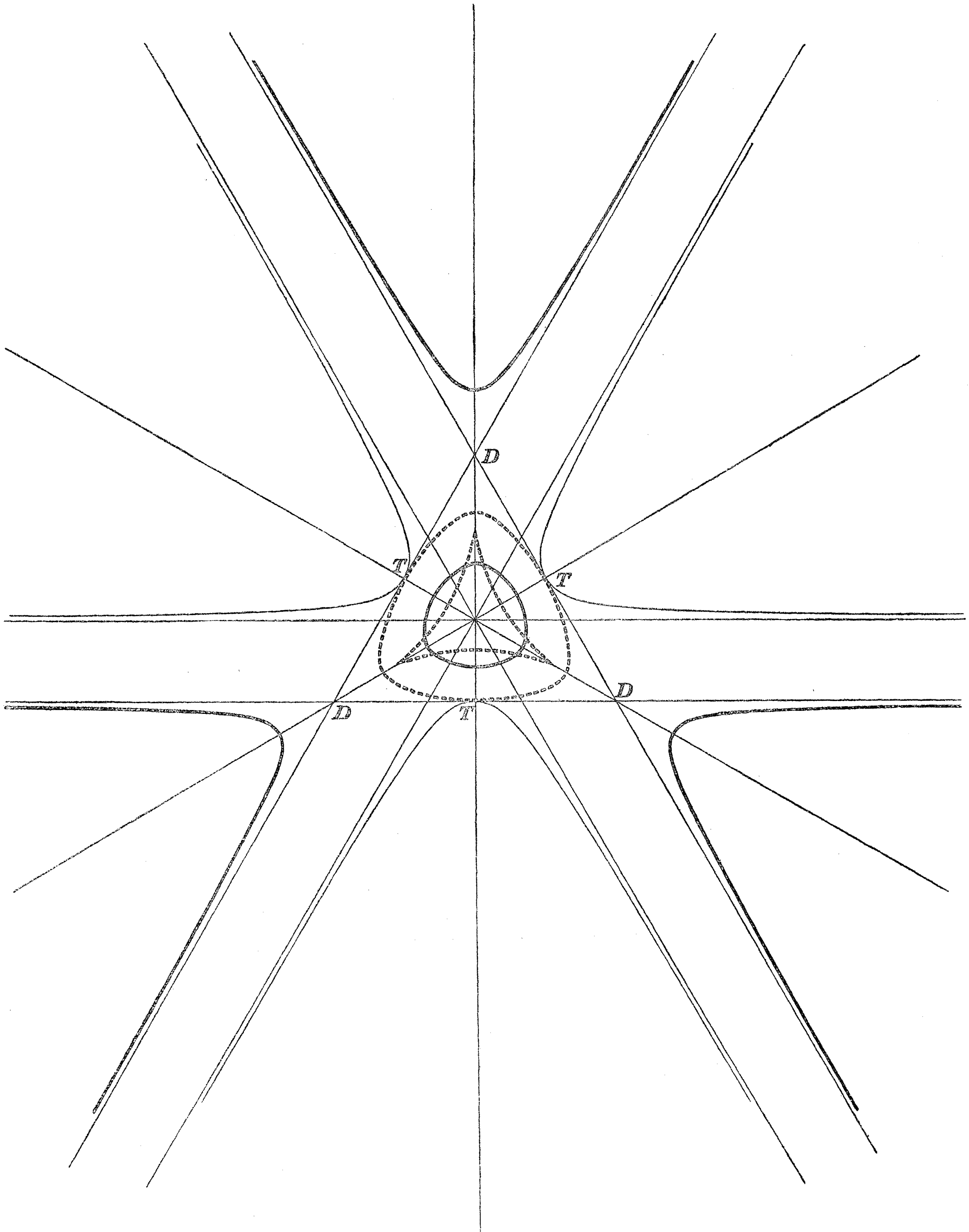

Fig. 8.

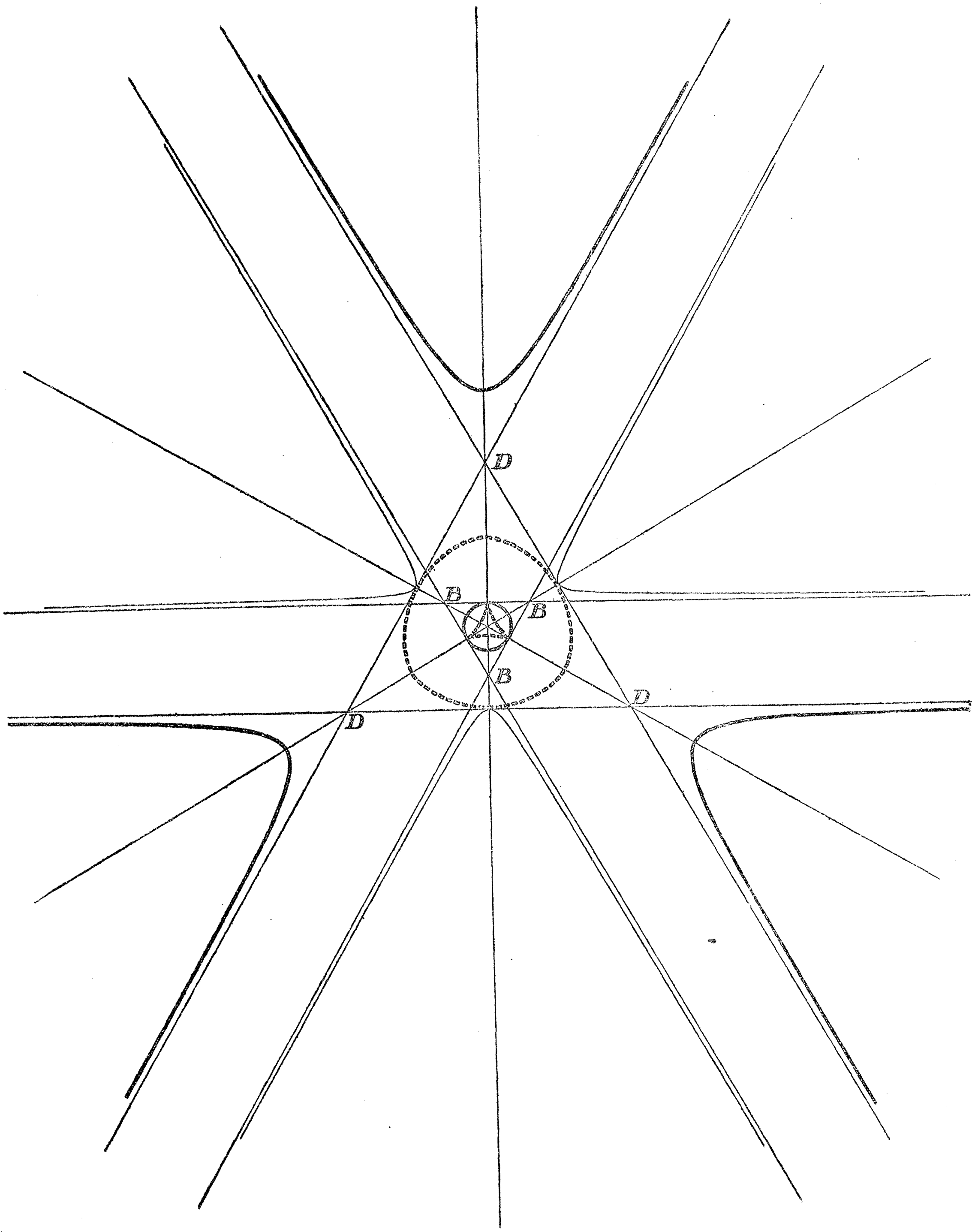

Fig. 9.

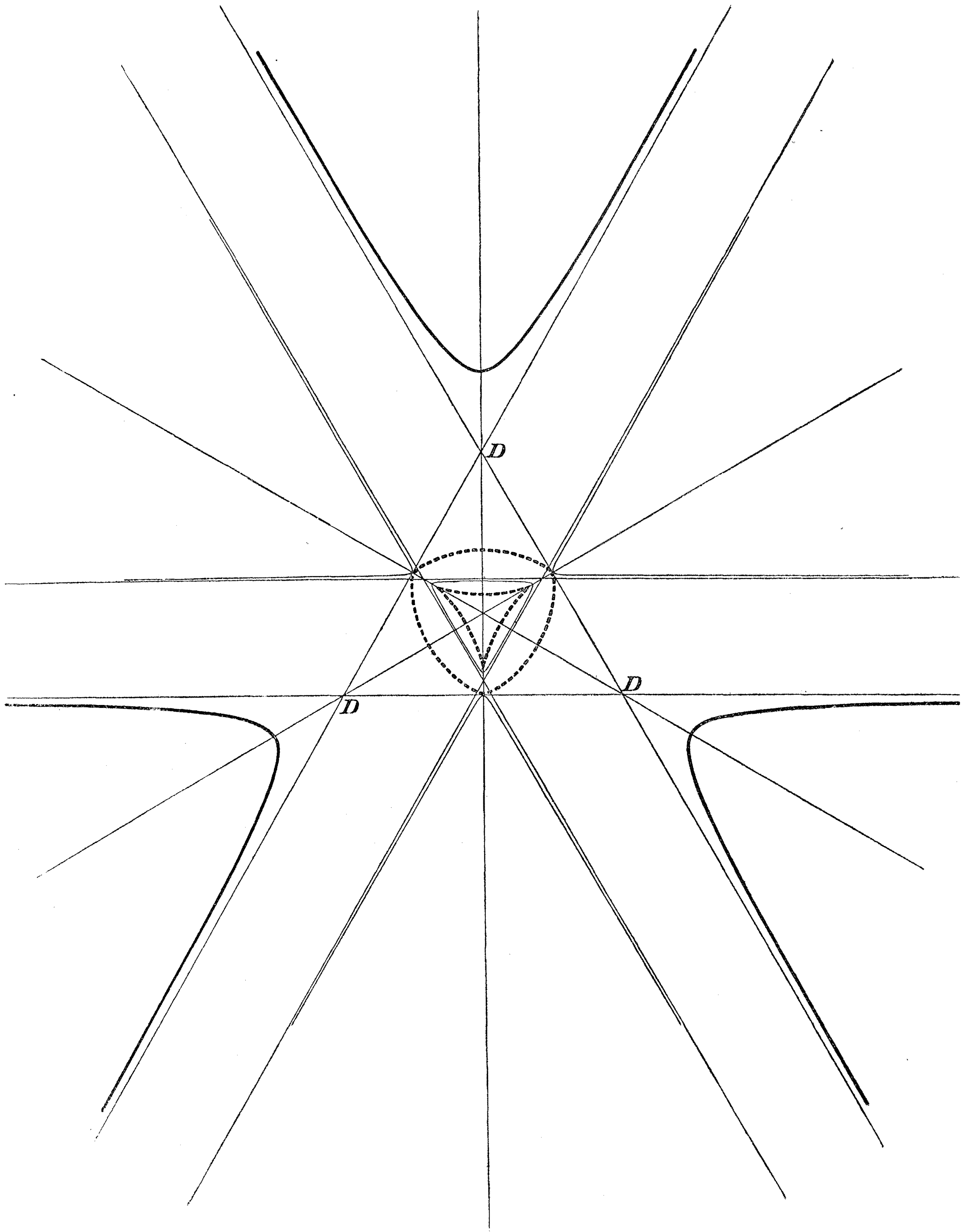

Fig. 10.

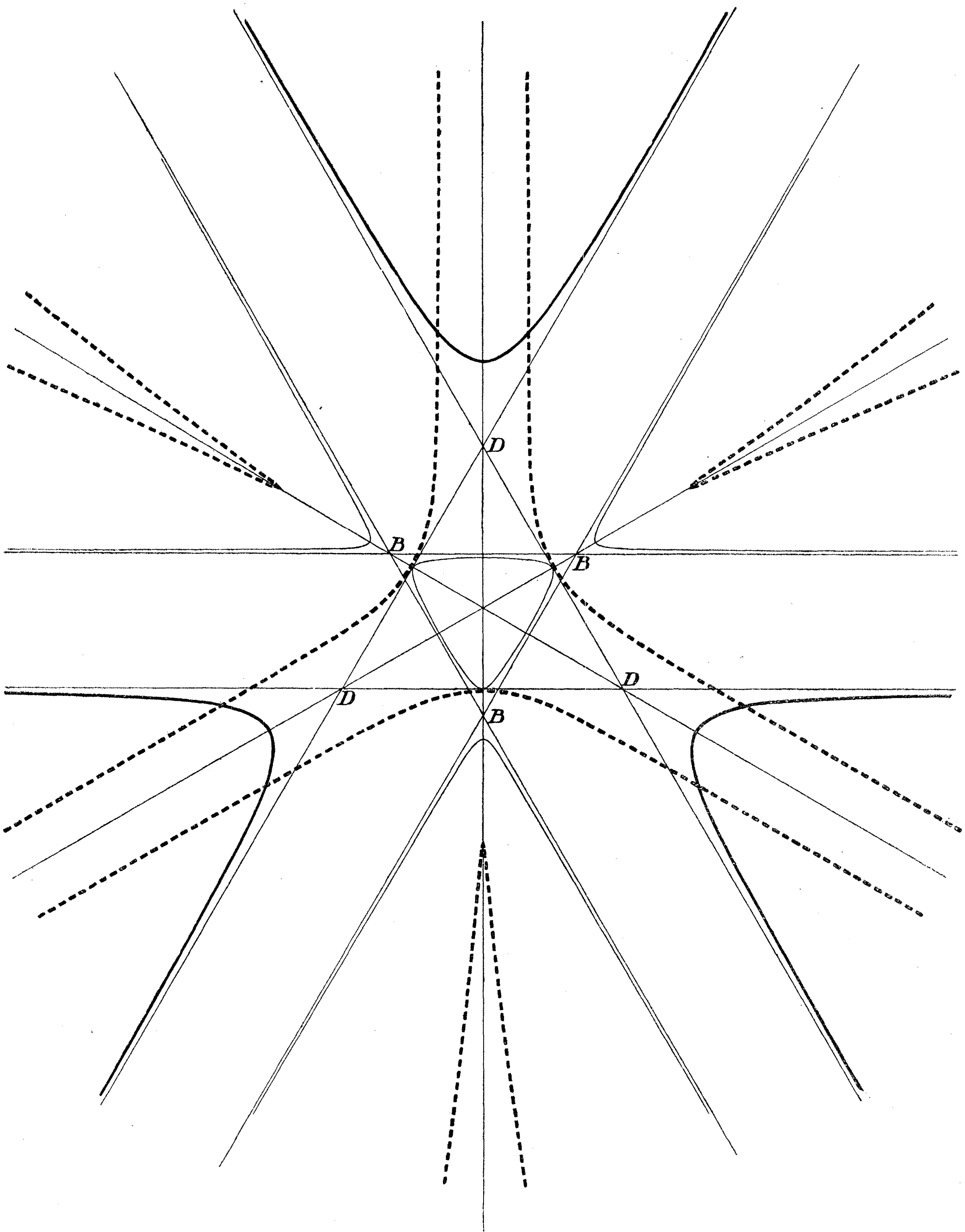

Fig. 11.

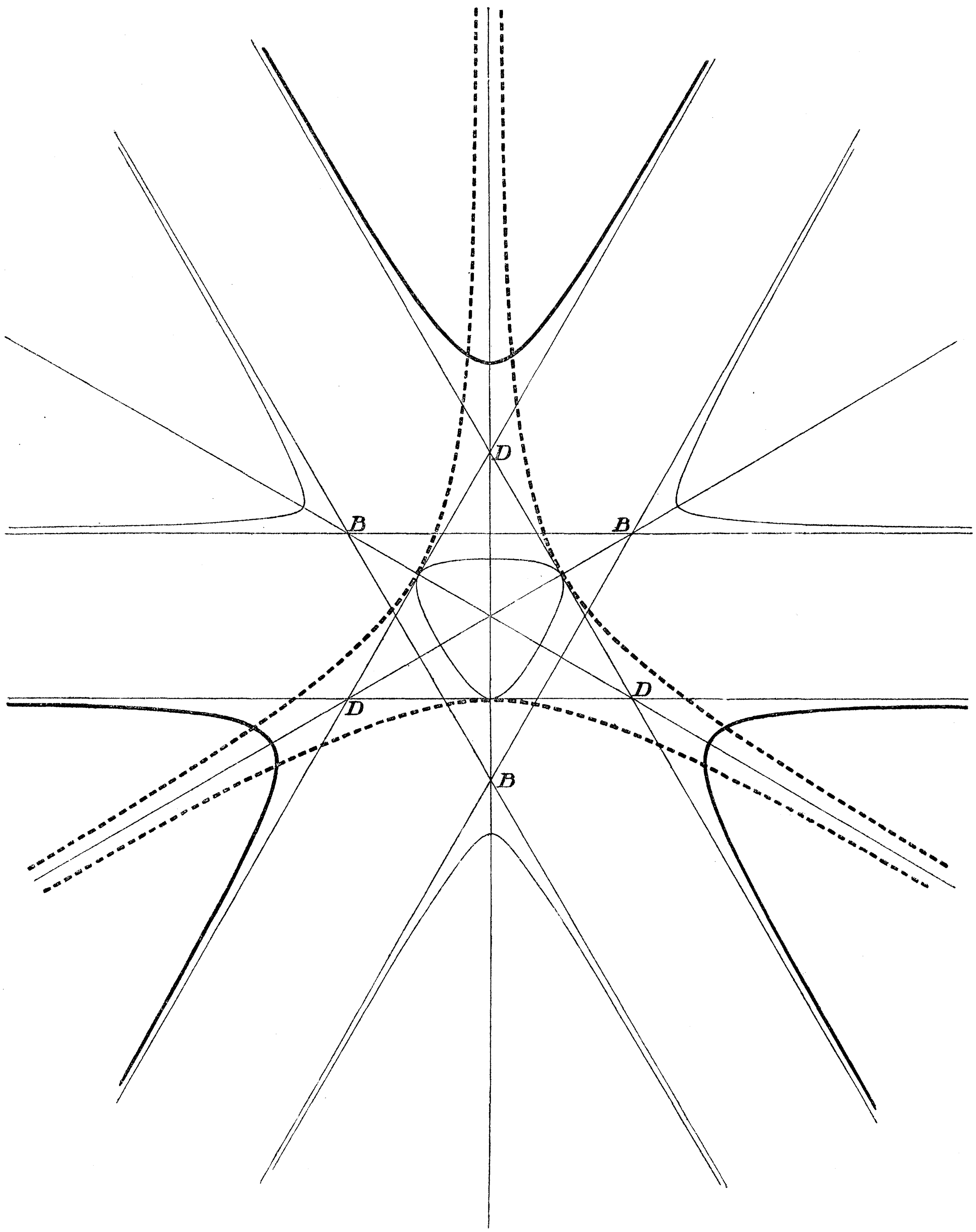

Fig. 12.

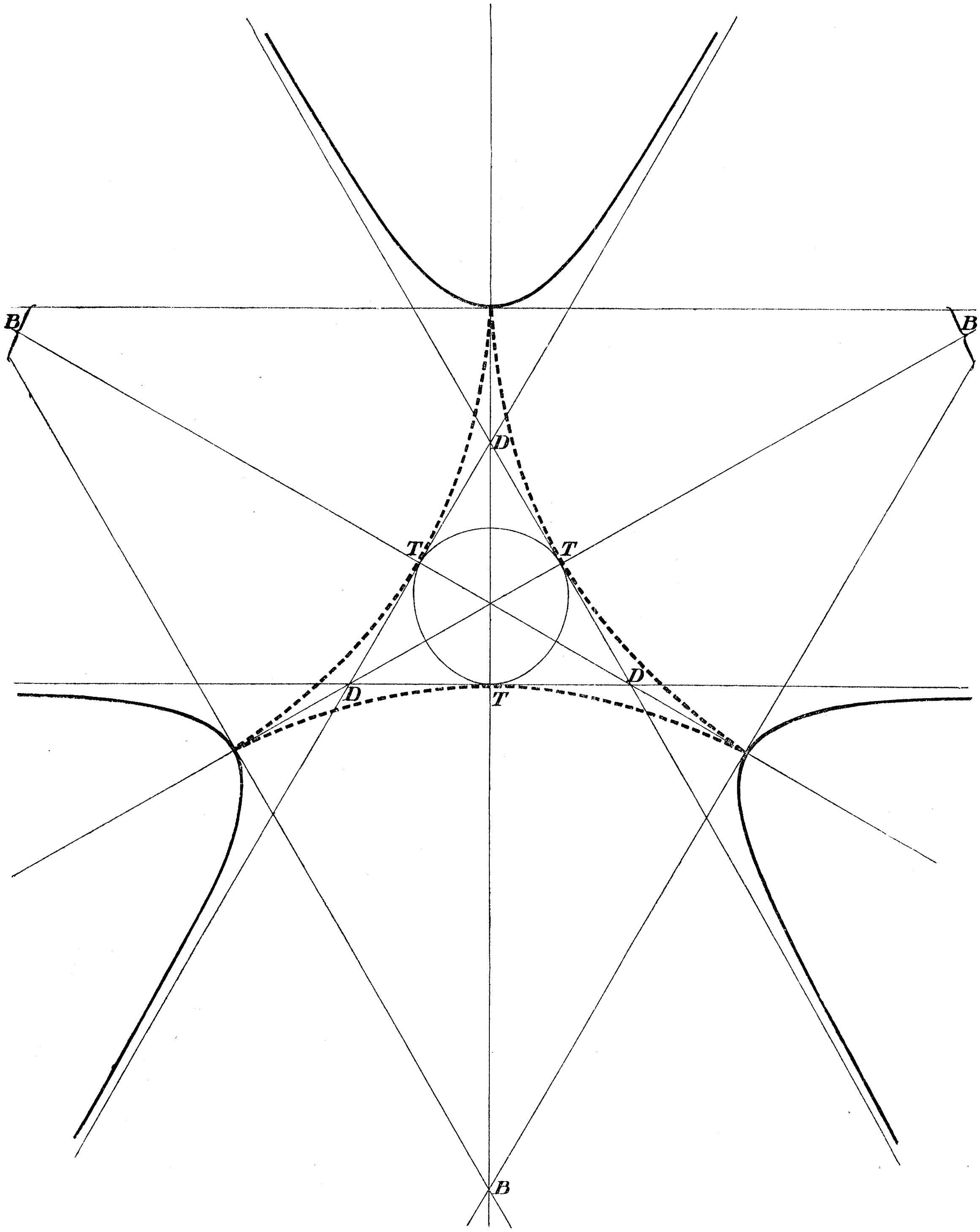

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Fig. 13.

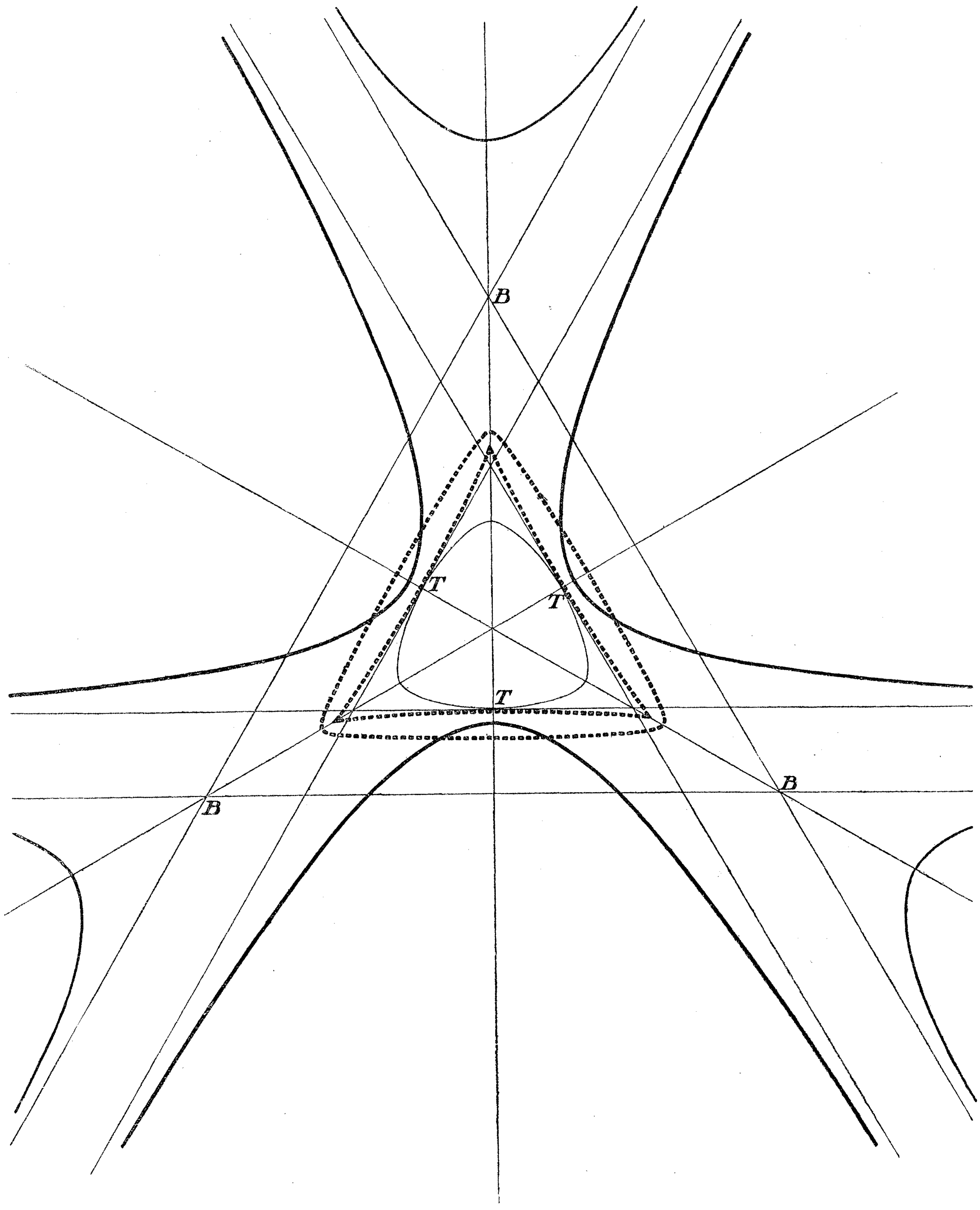

Fig. 14.